Pointwise strong and very strong approximation by Fourier series of integrable functions

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Abstract

We will present an estimation of the $H^q_{k_0,k_r}f$ and $H^{\lambda\varphi}_u f$ means as a approximation versions of the Totik type generalization(see [8, 9]) of the results of J. Marcinkiewicz and A. Zygmund in [7, 10]. As a measure of such approximations we will use the function constructed on the base of definition of the Gabisonia points [1]. Some results on the norm approximation will also given.

Key words: Pointwise approximation, Strong and very strong approximation

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[1]Key words: Strong approximation, rate of pointwise strong summability
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1 Introduction

Let L^p $(1 < p < \infty)$ $[resp.C]$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X = X^p$ where $X^p = L^p$ when $1 < p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(x)\|_{X^p} = \begin{cases} \left(\int_Q \mid f(x) \mid^p dx\right)^{1/p} & when\ 1 < p < \infty\ , \\ \sup_{x \in Q} \mid f(x) \mid & when\ \ p = \infty. \end{cases}$$

Consider the trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)$$

and denote by $S_k f$ the partial sums of Sf. Then,

$$H^q_{k_0,k_r}(x) := \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} |S_{k_\nu} f(x) - f(x)|^q \right\}^{\frac{1}{q}}, \qquad (q > 0).$$

where $0 \leq k_0 < k_1 < k_2 < ... < k_r \ \ (\geq r)$, and

$$H^{\lambda\varphi}_u f(x) := \left\{ \sum_{\nu=0}^{\infty} \lambda_\nu(u) \varphi(|S_\nu f(x) - f(x)|) \right\},$$

where (λ_ν) is a sequence of positive functions defined on the set having at least one limit point and a function $\varphi : [0, \infty) \to \mathbf{R}$.

As a measure of the above deviations we will use the pointwise characteristic, constructed on the base of definition of the Gabisonia points ($G_{p,s} - points$), introduced in [1] as follows

$$G_x f(\delta)_{p,s} := \left\{ \sum_{k=1}^{[\pi/\delta]} \left(\frac{1}{k\delta} \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p dt \right)^{s/p} \right\}^{1/s}$$

$$G^\circ_x f(\gamma)_{p,s} := \sup_{0<\delta\leq\gamma} \left\{ \sum_{k=1}^{[\pi/\delta]} \left(\frac{1}{k\delta} \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p dt \right)^{s/p} \right\}^{1/s}$$

and, constructed on the base of definition of the Lebesgue points ($L^p - points$), defined as usually

$$\begin{aligned} w_x f(\delta)_p \quad &: \ = \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p dt \right\}^{1/p}, \\ \text{where} \quad \varphi_x(t) \quad &: \ = f(x+t) + f(x-t) - 2f(x). \end{aligned}$$

We can observe that, for any $s > 0$ and $p \in [1, \infty)$

$$w_x f(\delta)_p \leq G_x f(\delta)_{p,s} \ ,$$

for $p \in [1, \infty)$ and $f \in C$

$$w_x f(\delta)_p \leq \omega_C f(\delta) .$$

By the Minkowski inequality, with $\widetilde{p} \geq s > p \geq 1$ for $f \in X^{\widetilde{p}}$,

$$\|G.f(\delta)_{p,s}\|_{X^{\widetilde{p}}} \leq \omega_{X^{\widetilde{p}}} f\left(\frac{|\log[\pi/\delta]|}{(\pi/\delta)^{\frac{1}{p}-\frac{1}{s}}}\right) \quad \text{(cf. [1])}$$

and

$$\|w_{\cdot\cdot} f(\delta)_p\|_{X^{\widetilde{p}}} \leq \omega_{X^{\widetilde{p}}} f(\delta) ,$$

where $\omega_X f$ is the modulus of continuity of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0<|h|\leq\delta} \|\varphi.(h)\|_X \quad .$$

It is well-known that $H^q_{0,r} f(x) -$ means tend to 0 (as $r \to \infty$) at the $L^p -$ *points* x of $f \in L^p$ $(1 < p \leq \infty)$ and at the $G_{1,s} -$ *points* x of $f \in L^1$ $(s > 1)$. These facts were proved as a generalization of the Fejér classical result on the convergence of the $(C, 1)$ -means of Fourier series by G. H. Hardy, J. E. Littlewood in [3].and by O. D. Gabisonia in [1]. In case L^1 and convergence almost everywhere the first results on this area belong to J. Marcinkiewicz [7] and A. Zygmund [10]. The estimates of $H^q_{0,r} f(x)$ -mean were obtained in [**?**, **?**, 5]. Here we present estimations of the $H^q_{k_0,k_r}(x)$ and $H^{\lambda\varphi}_v f(x)$ means as approximation versions of the Totik type (see [8, 9]) generalization of the result of O. D. Gabisonia [1]. We also give some corollaries on norm approximation.

By K we shall designate either an absolute constant or a constant depending on the some parameters, not necessarily the same of each occurrence. We shall write $I_1 \ll I_2$ if there exists a positive constant K , sometimes depended on some parameters, such that $I_1 \leq K I_2$.

2 Statement of the results

Let us consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2$ and let

$$L^p(w_x)_s = \left\{ f \in L^p : \ G_x f(\delta)_{p,s} \leq w_x(\delta) \quad , \text{ where } \ \delta > 0, \ \ s > p \geq 1 \right\}.$$

In the same way let

$$X^p(\omega)_s = \left\{ f \in X^p : \left\| G^\circ_\cdot f(\delta)_{1,s} \right\|_{X^p} \leq \omega(\delta) , \ \ \text{with a modulus of continuity } \ \omega \right\}$$

We start with theorems:

Theorem 1 *If* $f \in L^1(w_x)_s$ *and* $0 \leq k_0 < k_1 < k_2 < ... < k_r \quad (\geq r)$ *, then*

$$H^{q'}_{k_0,k_r}(x) \ll w_x\left(\frac{\pi}{k_0+1}\right)\log\frac{k_r+1}{r+1/2} \quad ,$$

where $0 < q' \leq q \quad (\geq 2)$ *such that* $\frac{1}{s}+\frac{1}{q}=1$.

Theorem 2 *If* $f \in X^p$ *and* $0 \leq k_0 < k_1 < k_2 < ... < k_r \quad (\geq r)$ *, then*

$$\left\| H^{q'}_{k_0,k_r} f(\cdot)\right\|_{X^p} \ll \omega\left(\frac{\pi}{k_0+1}\right)\log\frac{k_r+1}{r+1/2} \quad ,$$

where $0 < q' \leq q \quad (\geq 2)$ *such that* $\frac{1}{s}+\frac{1}{q}=1$.

Denoting

$$\Phi = \left\{ \begin{array}{c} \varphi : \varphi(0)=0, \quad \varphi \nearrow, \quad \varphi(2u) \ll \varphi(u) \quad \text{for } u \in (0,1) \\ \text{and} \quad \log\varphi(u) = O(u) \quad \text{as} \quad u \to \infty \end{array} \right\}$$

we can formulate the next theorems on the base of the before two.

Theorem 3 *If* $f \in L^1$, $\varphi \in \Phi$ *and* $\lambda_\nu(m) = \frac{1}{N_m+1}$ *for* $\nu = N_{m-2}+1, N_{m-2}+2, ..., N_m$ *and* $\lambda_\nu(m) = 0$ *otherwise, then*

$$H^{\lambda\varphi}_m f(x) \ll \varphi\left(w_x\left(\frac{\pi}{N_{m-2}+1}\right)\right) \quad ,$$

where $m = 1.2, ..$ *and* $s > 1$.

Theorem 4 *If* $f \in X^p$, $\varphi \in \Phi$ *and* $\lambda_\nu(m) = \frac{1}{N_m+1}$ *for* $\nu = N_{m-2}+1, N_{m-2}+2, ..., N_m$.*and* $\lambda_\nu(m) = 0$ *otherwise, then*

$$\left\| H^{\lambda\varphi}_m f(\cdot)\right\|_{X^p} \ll \varphi\left(\omega\left(\frac{\pi}{N_{m-2}+1}\right)\right) \quad ,$$

where $m = 1.2, ..$ *and* $s > 1$.

Let, as in the Leindler monograph [4] p.15,

$$\Lambda_\tau(N_m) = \left\{(\lambda_\nu) : \left(\frac{1}{N_m}\sum_{\nu=N_{m-2}+1}^{N_m}(\lambda_\nu)^\tau\right)^{1/\tau} \ll \left(\frac{1}{N_m}\sum_{\nu=N_{m-2}+1}^{N_m}\lambda_\nu\right)\right.$$
$$\text{for} \quad s \geq 1 \quad \text{and } N_m < N_{m+1} \text{ , } N_0 = 0, \quad N_{-1} = -1\Big\} .$$

Finally, we present very general results deduced from the above theorems.

Theorem 5 *If* $f \in L^1$ *then for* $(\lambda_\nu) \in \Lambda_\tau(N_m)$ *with* $\tau > 1$ *and for* $\varphi \in \Phi$, *we have*

$$H^{\lambda\varphi}_u f(x) \ll \sum_{m=1}^{\infty}\sum_{\nu=N_{m-2}+1}^{N_m}\lambda_\nu(u)\,\varphi\left(w_x\left(\frac{\pi}{N_{m-2}+1}\right)\right) \quad ,$$

for any real u *and* $s > 1$.

Theorem 6 *If $f \in X^p$ then, for $(\lambda_\nu) \in \Lambda_\tau(N_m)$ with $\tau > 1$ and for $\varphi \in \Phi$, we have*

$$\left\| H_u^{\lambda\varphi} f(\cdot) \right\|_{X^p} \ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu(u) \varphi\left(\omega\left(\frac{\pi}{N_{m-2}+1}\right)\right) \quad ,$$

for any real u and $s > 1$.

From these theorems we can derive the following corollary.

Corollary 1 *If we additionally suppose that $\lim_{u \to u_0} \lambda_\nu(u) = 0$ for all ν and that $\sum_\nu^\infty \lambda_\nu(u)$ converges, then we have*

$$\lim_{u \to u_0} H_u^{\lambda\varphi} f(x) = 0$$

at every $G_{1,s}-points$ x of the function f, and

$$\lim_{u \to u_0} \left\| H_u^{\lambda\varphi} f(\cdot) \right\|_{L^p} = 0.$$

for any real $s > 1$.

Remark 1 *We can observe that in the light of the Gabisonia result [2] our pointwise results remain true for $f \in L^p$ $(p > 1)$, since every $L^p-point$ of the function f is its $G_{p,s}-point$.*

3 Auxiliary results

At the begin we present some lemmas on pointwise characteristics.

Lemma 1 *(Property 1 [5]) If $f \in L^p$ $(p \geq 1)$ and $\lambda, \beta > 0$, then*

$$\left\{ \lambda^\beta \int_\lambda^\pi t^{-(\beta+1)} \left|\varphi_x(t)\right|^p dt \right\}^{1/\beta} \ll G_x f(\lambda)_{p,s}$$

with $s > p$ such that $s(1-\beta) < p$.

Lemma 2 *(Property 2 [5]) If $f \in L^p$ $(p \geq 1)$ and $\lambda, \beta > 0$, then*

$$G_x f(2\lambda)_{p,s} \leq 2^{1/p-1/s} G_x f(\lambda)_{p,s}$$

with $s > p$.

Lemma 3 *If $f \in L^p$ $(p \geq 1)$, then*

$$\begin{aligned} &\left\{ \frac{1}{\delta} \int_0^\delta \left|\varphi_x(t+\gamma) - \varphi_x(t)\right|^p dt \right\}^{1/p} \\ \leq \quad &\left(2^{1/p} + 4^{1/p}\right) w_x f(2\delta) \leq \left(2^{1/p} + 4^{1/p}\right) G_x f(\delta)_{p,s} \end{aligned}$$

for any positive $\gamma \leq \delta$ and $1 \leq p < s$.

Proof. Since $\gamma \leq \delta$ we have

$$\begin{aligned}
&\left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t \pm \gamma \right) - \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
\leq\ & \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_{\pm\gamma}^{\delta \pm \gamma} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
\leq\ & \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_{-2\delta}^{2\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
\leq\ & \left\{ \frac{2}{2\delta} \int_0^{2\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} + \left\{ \frac{2}{\delta} \int_0^{2\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
=\ & \left(2^{1/p} + 4^{1/p} \right) \left\{ \frac{1}{2\delta} \int_0^{2\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p}
\end{aligned}$$

and our inequalities are evident. ■

Under the notation

$$\Phi_x f \left(\delta, \gamma \right) := \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} \varphi_x \left(t \right) dt, \qquad W_x f \left(\delta, \gamma \right)_p := \left[\frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} \left| \varphi_x \left(t \right) \right|^p dt \right]^{1/p}$$

we can formulate a lemma.

Lemma 4 *If $f \in L^p \quad (p \geq 1)$, then*

$$\left| \Phi_x f \left(\delta, \gamma \right) \right| \leq W_x f \left(\delta, \gamma \right)_p \ll w_x f \left(2\delta \right)$$

for any positive $\gamma \leq \delta$.

Proof. The first inequality is evidence, then we prove the second one only.

If $f \in L^p$, then

$$\begin{aligned}
&\left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t + \gamma \right) \right|^p dt \right\}^{1/p} - \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
\leq\ & \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t + \gamma \right) - \varphi_x \left(t \right) \right|^p dt \right\}^{1/p}
\end{aligned}$$

whence

$$\begin{aligned}
&\left\{ \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} \\
\leq\ & \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t + \gamma \right) - \varphi_x \left(t \right) \right|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x \left(t \right) \right|^p dt \right\}^{1/p}
\end{aligned}$$

and by the previous lemma our second relation follows. ■

We will also need the inequalities for norms.

Lemma 5 *If $f \in L^p \quad (p \geq 1)$, then*

$$\left\| \Phi .f\left(\delta ,\gamma \right) \right\| _{L^{p}}\leq \left\| W.f\left(\delta ,\gamma \right) _{p}\right\| _{L^{p}}\leq 2\omega _{L^{p}}f\left(\delta +\gamma \right)$$

and

$$\left\| \left[\frac{1}{\delta }\int_{0}^{\delta }\left| \varphi .\left(t\right) -\varphi .\left(t\pm \gamma \right) \right| ^{p}dt\right] ^{1/p}\right\| _{L^{p}}\leq 2\omega _{L^{p}}f\left(\gamma \right) \quad ,$$

for any positive γ and δ.

Proof. If $f \in L^p$, then, by monotonicity of the norm as a functional and by the above Lemma,

$$\left\| \Phi .f\left(\delta ,\gamma \right) \right\| _{L^{p}}\leq \left\| W.f\left(\delta ,\gamma \right) _{p}\right\| _{L^{p}}$$

and consequently

$$\begin{aligned}
\left\| w.f\left(\delta ,\gamma \right) _{p}\right\| _{L^{p}} &= \left\{ \int_{-\pi }^{\pi }\left[\frac{1}{\delta }\int_{\gamma }^{\gamma +\delta }\left| \varphi _{x}\left(t\right) \right| ^{p}dt\right] dx\right\} ^{1/p} \\
&= \left\{ \frac{1}{\delta }\int_{\gamma }^{\gamma +\delta }\left[\int_{-\pi }^{\pi }\left| \varphi _{x}\left(t\right) \right| ^{p}dx\right] dt\right\} ^{1/p} \\
&\leq \left\{ \frac{1}{\delta }\int_{\gamma }^{\gamma +\delta }\left[2\omega _{L^{p}}f\left(t\right) \right] ^{p}dt\right\} ^{1/p} \\
&\leq 2\omega _{L^{p}}f\left(\delta +\gamma \right) ,
\end{aligned}$$

whence our first result follows.

In the next one we will change order of integration, whence

$$\begin{aligned}
&\left\| \left[\frac{1}{\delta }\int_{0}^{\delta }\left| \varphi .\left(t\right) -\varphi .\left(t\pm \gamma \right) \right| ^{p}dt\right] ^{1/p}\right\| _{L^{p}} \\
\leq &\left\{ \frac{1}{\delta }\int_{0}^{\delta }\left[\int_{-\pi }^{\pi }\left| \varphi _{x}\left(t\right) -\varphi _{x}\left(t\pm \gamma \right) \right| ^{p}dx\right] dt\right\} ^{1/p} \\
\leq &\left\{ \frac{1}{\delta }\int_{0}^{\delta }\left[\int_{-\pi }^{\pi }\left(\left| f\left(x+t\right) -f\left(x+t\pm \gamma \right) \right| \right. \right. \right. \\
&\left. \left. \left. +\ \left| f\left(x-t\right) -f\left(x-t\mp \gamma \right) \right| \right) ^{p}dx\right] dt\right\} ^{1/p} \\
\leq &\left\{ \frac{1}{\delta }\int_{0}^{\delta }\left[2\omega _{L^{p}}f\left(\gamma \right) \right] ^{p}dt\right\} ^{1/p}=2\omega _{L^{p}}f\left(\gamma \right)
\end{aligned}$$

and thus our proof is complete. ■

In the sequel we will also need some another lemmas with the next notions. Let

$$\Psi_x f(\delta,\gamma)_p := \left\{ \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} |\varphi_x(t)|^p \, dt \right\}^{1/p},$$

then we have

Lemma 6 *If $f \in L^p \quad (p \geq 1)$, then*

$$\Psi_x f(\delta,\gamma)_p \ll G_x f(\delta)_{p,s}$$

for any positive $\delta \leq \gamma$ such that $\gamma+\delta \leq \pi \quad$ and $1 \leq p < s$.

Proof. There exists a natural k' such that $(k'-1)\,\delta \leq \gamma+\delta \leq k'\delta$. Then

$$\begin{aligned}
\Psi_x f(\delta,\gamma)_p &\ll \left(\frac{1}{k'\delta} \int_{(k'-2)\delta}^{k'\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} \\
&\ll \left(\frac{1}{k'\delta} \int_{(k'-1)\delta}^{k'\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} + \left(\frac{1}{(k'-1)\,\delta} \int_{(k'-2)\delta}^{(k'-1)\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} \\
&\ll \left\{ \sum_{k=1}^{[\pi/\delta]} \left(\frac{1}{k\delta} \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p \, dt \right)^{s/p} \right\}^{1/s} = G_x f(\delta)_{p,s}
\end{aligned}$$

and our estimate is proved.

Lemma 7 *If $f \in L^p \quad (p \geq 1)$, then*

$$\|\Psi_{\cdot} f(\delta,\gamma)_1\|_{L^p} \ll \omega_{L^p} f(\delta)$$

for any positive $\delta \leq \gamma$ such that $\gamma+\delta \leq \pi$.

Proof. Easy calculation gives

$$\begin{aligned}
\|\Psi_{\cdot} f(\delta,\gamma)_1\|_{L^p} &\ll \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} \omega_{L^p} f(t)\, dt \ll \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} \omega_{L^p} f(\gamma+\delta)\, dt \\
&\ll \frac{\omega_{L^p} f(\gamma)}{\gamma} \int_{\gamma}^{\gamma+\delta} dt = \delta \frac{\omega_{L^p} f(\gamma)}{\gamma} \ll \delta \frac{\omega_{L^p} f(\delta)}{\delta}
\end{aligned}$$

and our Lemma is proved. ■ ■

4 Proofs of the results

We only prove Theorems 1, 3 and 5 because in the remain proofs we have to use Lemma 5 and Lemma 7 instead of Lemmas 3, 4 and 6.

4.1 Proof of Theorem 1

Let

$$\begin{aligned} H^{q}_{k_0,k_r}(x) &= \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\ &\leq A_r + B_r + C_r, \end{aligned}$$

where

$$A_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^{2\delta} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q},$$

$$B_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q},$$

$$C_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\gamma}^{\pi} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q},$$

with $D_{k_\nu}(t) = \frac{\sin\left(\left(k_\nu + \frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}}$, $\delta = \delta_\nu$ and $\gamma = \gamma_r^2/\delta_\nu$, putting $\delta_\nu = \frac{\pi}{k_\nu+1/2}$, $\gamma_r = \frac{\pi}{r+1/2}$. In the case $\gamma \geq \pi/2$ we will have $C_r \equiv 0$. At the begin

$$\begin{aligned} A_r &\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{k_\nu+1}{\pi} \int_0^{2\delta} |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} \\ &\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[4\frac{k_\nu+1/2}{2\pi} \int_0^{2\delta_\nu} |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} \\ &\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} [4w_x f(2\delta_\nu)_1]^q \right\}^{1/q} \\ &\leq 4w_x(2\delta_0) \leq 8w_x(\delta_0). \end{aligned}$$

The terms B_{k_r} and C_{k_r} we estimate by the Totik method [9].and its modification from [6] We divide the term B_r into the two parts

$$\begin{aligned} B_r &= \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\ &\leq \left\{ \frac{1}{r+1} \left(\sum_{\nu=0}^{\nu_0-1} + \sum_{\nu=\nu_0}^{r} \right) \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{\nu_0-1} \left| \frac{1}{\pi} \int_{2\gamma}^{2\delta} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\
&\quad + \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{\nu_0-1} \left[\frac{k_\nu+1}{\pi} \int_{2\gamma}^{2\delta} |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} + B_{r,\nu_0} \\
&\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{4}{2\delta_\nu} \int_{0}^{2\delta_\nu} |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} + B_{r,\nu_0}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} [4 w_x f(2\delta_\nu)_1]^q \right\}^{1/q} + B_{r,\nu_0} \\
&\leq 8 w_x(\delta_0) + B_{r,\nu_0} \ ,
\end{aligned}
$$

where the index ν_0 is such that $k_{\nu_0-1} < r \leq k_{\nu_0}$ $(\delta_{\nu_0} \leq \gamma_r < \delta_{\nu_0-1}$ with $k_{-1} = 0)$. Next the term B_{r,ν_0}. we divide into the three parts.

$$
\begin{aligned}
& B_{r,\nu_0}. \\
&= \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\
&= \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \left(\int_{2\delta_\nu}^{2\gamma} + \int_{\delta_\nu}^{2\gamma-\delta_\nu} + \int_{2\gamma-\delta_\nu}^{2\gamma} - \int_{\delta_\nu}^{2\delta_\nu} \right) \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\
&\leq B^1_{r,\nu_0} + B^2_{r,\nu_0} + B^3_{r,\nu_0},
\end{aligned}
$$

where the first term

$$
\begin{aligned}
& B^1_{r,\nu_0} \\
&= \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \left(\int_{2\delta_\nu}^{2\gamma} + \int_{\delta_\nu}^{2\gamma-\delta_\nu} \right) \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\
&= \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} [\varphi_x(t) D_{k_\nu}(t) + \varphi_x(t-\delta_\nu) D_{k_\nu}(t-\delta_\nu)]\, dt \right|^q \right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} (\varphi_x(t) - \varphi_x(t-\delta_\nu)) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\
&\quad + \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^{r} \left| \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} \varphi_x(t-\delta_\nu) (D_{k_\nu}(t) + D_{k_\nu}(t-\delta_\nu))\, dt \right|^q \right\}^{1/q}.
\end{aligned}
$$

Using the partial integration we obtain

$$
\begin{aligned}
& B^1_{r,\nu_0} \\
\leq\ & \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma}\frac{d}{dt}\left[\int_0^t\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}du\right]\frac{1}{2\sin\frac{t}{2}}dt\right|^q\right\}^{1/q} \\
& +\frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma}\varphi_x(t-\delta_\nu)\left(\frac{1}{2\sin\frac{t}{2}}-\frac{1}{2\sin\frac{t-\delta_\nu}{2}}\right)\sin\frac{(2k_\nu+1)\,t}{2}dt\right|^q\right\}^{1/q} \\
\ll\ & \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\left[\int_0^t\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}du\frac{1}{2\sin\frac{t}{2}}\right]_{t=2\delta_\nu}^{2\gamma}\right.\right. \\
& \left.\left.+\ \frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}\left[\int_0^t\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}du\right]\frac{\cos\frac{t}{2}}{\left(2\sin\frac{t}{2}\right)^2}dt\right|^q\right\}^{1/q} \\
& +\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\delta_\nu\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}\frac{\left|\varphi_x(t-\delta_\nu)\right|}{t^2}dt\right|^q\right\}^{1/q} \\
\ll\ & \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\left|\frac{1}{\pi}\int_0^{2\gamma}\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}du\frac{1}{2\sin\frac{2\gamma}{2}}\right|\right.\right. \\
& +\left|\frac{1}{\pi}\int_0^{2\delta_\nu}\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}du\frac{1}{2\sin\frac{2\delta_\nu}{2}}\right| \\
& \left.\left.+\ \frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma}\left[\int_0^t\left|\left(\varphi_x(u)-\varphi_x(u-\delta_\nu)\right)\sin\frac{(2k_\nu+1)\,u}{2}\right|du\right]\frac{\pi^2}{(2t)^2}dt\right]^q\right\}^{1/q} \\
& +\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\delta_\nu\frac{1}{\pi}\int_{\delta_\nu}^{2\gamma-\delta_\nu}\frac{\left|\varphi_x(t)\right|}{(t+\delta_\nu)^2}dt\right]^q\right\}^{1/q},
\end{aligned}
$$

and applying Lemmas 1,2,3 we have

$$
\begin{aligned}
& B^1_{r,\nu_0} \\
\ll\ & \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\frac{1}{8\gamma}\int_0^{2\gamma}\left|\varphi_x(u)-\varphi_x(u-\delta_\nu)\right|du\right.\right. \\
& +\ \frac{1}{4\delta_\nu}\int_0^{2\delta_\nu}\left|\varphi_x(u)-\varphi_x(u-\delta_\nu)\right|du \\
& \left.\left.+\ \frac{\pi}{8}\int_{2\delta_\nu}^{2\gamma}\left(\frac{1}{t^2}\int_0^t\left|\varphi_x(u)-\varphi_x(u-\delta_\nu)\right|du\right)dt\right]^q\right\}^{1/q} \\
& +\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\delta_\nu\int_{\delta_\nu}^{\pi}\frac{\left|\varphi_x(t)\right|}{t^2}dt\right]^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\ll w_x(\delta_0) \\
&\quad +\frac{\pi}{8}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\int_{2\delta_\nu}^{2\gamma}\frac{1}{t}w_x(\delta_\nu)\,dt\right]^q\right\}^{1/q} \\
&\quad +\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[\delta_\nu\sum_{\mu=0}^{k_\nu}w_x f\left(\frac{\pi}{\mu+1}\right)_1\right]^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\ll w_x(\delta_0)+Kw_x(\delta_0)\log\frac{\gamma}{\delta_r}+K\delta_0\sum_{\mu=0}^{k_0}w_x\left(\frac{\pi}{\mu+1}\right) \\
&\leq Kw_x(\delta_0)\left(1+\log\frac{k_r+1/2}{r+1/2}\right).
\end{aligned}
$$

Consequently, by Lemma 4,

$$
\begin{aligned}
B_{r,\nu_0}^2 &= \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{2\gamma-\delta_\nu}^{2\gamma}\varphi_x(t)\,D_{k_\nu}(t)\,dt\right|^q\right\}^{1/q} \\
&\leq \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{2\gamma-\delta_\nu}^{2\gamma}|\varphi_x(t)|\,\frac{\pi}{2t}dt\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{2\gamma-\delta_\nu}^{2\gamma}|\varphi_x(t)|\,\frac{\pi}{2t}dt\right|^q\right\}^{1/q} \\
&\leq \frac{1}{4}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\int_{2\gamma-\delta_\nu}^{2\gamma}\frac{d}{dt}\left(\int_0^t|\varphi_x(u)|\,du\right)\frac{dt}{t}\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{4}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\left[\frac{1}{t}\int_0^t|\varphi_x(u)|\,du\right]_{t=2\gamma-\delta_\nu}^{t=2\gamma}+\int_{2\gamma-\delta_\nu}^{2\gamma}\frac{w_x(t)}{t}dt\right|^q\right\}^{1/q} \\
&\ll \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{2\gamma}\int_0^{2\gamma}|\varphi_x(u)|\,du-\frac{1}{2\gamma-\delta_\nu}\int_0^{2\gamma-\delta_\nu}|\varphi_x(u)|\,du\right.\right. \\
&\quad \left.\left.+\ \frac{w_x(2\gamma-\delta_\nu)}{2\gamma-\delta_\nu}\int_{2\gamma-\delta_\nu}^{2\gamma}dt\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\ll \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{2\gamma-\delta_\nu}\int_0^{2\gamma}\left[|\varphi_x(u)|-|\varphi_x(u-\delta_\nu)|\right]du\right.\right. \\
&\quad \left.\left.+\ \frac{1}{2\gamma-\delta_\nu}\int_0^{\delta_\nu}|\varphi_x(u-\delta_\nu)|\,du+\frac{w_x(\delta_\nu)}{\delta_\nu}\delta_\nu\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\ll \left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\gamma}\int_0^{2\gamma}\left[\left|\varphi_x(u)-\varphi_x(u-\delta_\nu)\right|\right]du\right.\right. \\
&\quad \left.\left.+\frac{1}{\delta_\nu}\int_{-\delta_\nu}^{0}\left|\varphi_x(u)\right|du+w_x(\delta_\nu)\right|^q\right\}^{1/q} \\
&\ll w_x(\delta_0)
\end{aligned}
$$

and

$$
\begin{aligned}
B_{r,\nu_0}^3 &= \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{\delta_\nu}^{2\delta_\nu}\varphi_x(t)\,D_{k_\nu}(t)\,dt\right|^q\right\}^{1/q} \\
&\le \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left|\frac{1}{\pi}\int_{\delta_\nu}^{2\delta_\nu}\left|\varphi_x(t)\right|\frac{\pi}{2t}dt\right|^q\right\}^{1/q} \\
&\le \frac{1}{2}\left\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\left[w_xf(2\delta_\nu)\right]^q\right\}^{1/q}\ll w_x(\delta_0)\,.
\end{aligned}
$$

Thus

$$
B_r\ll w_x(\delta_0)\left(1+\log\frac{k_r+1}{r+1/2}\right).
$$

Finally we estimate the term C_r dividing it into the two parts.

$$
\begin{aligned}
& C_r \\
= & \left\{\frac{1}{r+1}\sum_{\nu=0}^{r}\left|\frac{1}{\pi}\int_{2\gamma}^{\pi}\varphi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sin\left(\left(k_\nu+\frac{1}{2}\right)t\right)dt\right|^q\right\}^{1/q} \\
\le & \left\{\frac{1}{r+1}\sum_{\nu=0}^{r}\left|\frac{1}{\pi}\int_{2\gamma}^{\pi}\left[\frac{\Phi_xf(\delta_0,t)-\varphi_x(t)}{2\sin\frac{t}{2}}\right]\sin\left(\left(k_\nu+\frac{1}{2}\right)t\right)dt\right|^q\right\}^{1/q} \\
& +\left\{\frac{1}{r+1}\sum_{\nu=0}^{r}\left|\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{\Phi_xf(\delta_0,t)}{2\sin\frac{t}{2}}\sin\left(\left(k_\nu+\frac{1}{2}\right)t\right)dt\right|^q\right\}^{1/q} \\
= & C_r^1+C_r^2.
\end{aligned}
$$

Integrating by parts and applying Lemma 4 we obtain

$$
\begin{aligned}
C_r^1 &\le \frac{1}{\delta_0}\int_0^{\delta_0}\left[\int_{2\gamma}^{\pi}\frac{\left|\varphi_x(u+t)-\varphi_x(t)\right|}{t}dt\right]du \\
&= \frac{1}{\delta_0}\int_0^{\delta_0}\left[\int_{2\gamma}^{\pi}\frac{1}{t}\frac{d}{dt}\left(\int_0^t\left|\varphi_x(u+v)-\varphi_x(v)\right|dv\right)dt\right]du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\delta_0}\int_0^{\delta_0}\left\{\left[\frac{1}{t}\int_0^t \left|\varphi_x(u+v)-\varphi_x(v)\right| dv\right]_{t=2\gamma}^{\pi}\right.\\
&\quad \left.+\int_{2\gamma}^{\pi}\left(\frac{1}{t^2}\int_0^t \left|\varphi_x(u+v)-\varphi_x(v)\right| dv\right)dt\right\} du
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\delta_0}\int_0^{\delta_0} w_x(u)\,du+\frac{1}{\delta_0}\int_0^{\delta_0}\left\{\int_{2\gamma}^{\pi}\frac{1}{t}w_x(u)\,dt\right\}du\\
&\leq w_x(\delta_0)+w_x(\delta_0)\int_{2\gamma}^{\pi}\frac{1}{t}dt\\
&\leq w_x(\delta_0)(1+\log\pi-\log\gamma)\\
&\leq w_x(\delta_0)\left(1+\log\frac{k_r+1}{r+1}\right)
\end{aligned}
$$

and additionally by Lemma 6

$$
\begin{aligned}
&C_r^2\\
&= \frac{1}{2(r+1)^{1/q}}\left\{\sum_{\nu=0}^{r}\left|\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{\Phi_x f(\delta_0,t)}{2\sin\frac{t}{2}}\frac{d}{dt}\left(\frac{\cos\left(\left(k_\nu+\frac{1}{2}\right)t\right)}{k_\nu+\frac{1}{2}}\right)dt\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi(r+1)^{1/q}}\left\{\sum_{\nu=0}^{r}\left|\left[\frac{\Phi_x f(\delta_0,t)}{2\sin\frac{t}{2}}\frac{\cos\left(\left(k_\nu+\frac{1}{2}\right)t\right)}{k_\nu+\frac{1}{2}}\right]_{2\gamma}^{\pi}\right.\right.\\
&\quad \left.\left.-\int_{2\gamma}^{\pi}\frac{d}{dt}\left(\frac{\Phi_x f(\delta_0,t)}{2\sin\frac{t}{2}}\right)\frac{\cos\left(\left(k_\nu+\frac{1}{2}\right)t\right)}{k_\nu+\frac{1}{2}}dt\right|^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi(r+1)^{1/q}}\left\{\sum_{\nu=0}^{r}\left[\left|\frac{\Phi_x f(\delta_0,2\gamma)}{2\sin\gamma}\frac{\cos\left(\left(k_\nu+\frac{1}{2}\right)2\gamma\right)}{k_\nu+\frac{1}{2}}\right|\right.\right.\\
&\quad \left.\left.+\left|\int_{2\gamma}^{\pi}\frac{d}{dt}\left(\frac{\Phi_x f(\delta_0,t)}{2\sin\frac{t}{2}}\right)\frac{\cos\left(\left(k_\nu+\frac{1}{2}\right)t\right)}{k_\nu+\frac{1}{2}}dt\right|\right]^q\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\left|\Phi_x f(\delta_0,2\gamma)\right|}{\gamma(k_0+1)}+\frac{1}{k_0+1}\int_{2\gamma}^{\pi}\left|\frac{d}{dt}\left(\frac{\Phi_x f(\delta_0,t)}{2\sin\frac{t}{2}}\right)\right|dt\\
&\leq \frac{1}{\gamma(k_0+1)}\frac{1}{\delta_0}\int_0^{\delta_0}\left|\varphi_x(u+2\gamma)\right|du+\delta_0\int_{2\gamma}^{\pi}\frac{\left|\varphi_x(\delta_0+t)-\varphi_x(t)\right|}{\delta_0 t}dt\\
&\quad +\frac{1}{\delta_0}\int_0^{\delta_0}\left(\delta_0\int_{2\gamma}^{\pi}\frac{\left|\varphi_x(u+t)\right|}{t^2}dt\right)du
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \left|\Psi_x f\left(\delta_0, 2\gamma\right)\right| + \int_{2\gamma}^{\pi} \frac{1}{t} \frac{d}{dt}\left(\int_0^t \left|\varphi_x\left(\delta_0+u\right)-\varphi_x\left(u\right)\right| du\right) dt \\
&\quad + \frac{1}{\delta_0}\int_0^{\delta_0}\left(\delta_0 \int_{2\gamma}^{\pi} \frac{\left|\varphi_x\left(u+t\right)\right|}{t^2} dt\right) du \\
&\leq \quad G_x f\left(\delta_0\right)_{1,s} + \left[\frac{1}{t}\int_0^t \left|\varphi_x\left(\delta_0+u\right)-\varphi_x\left(u\right)\right| du\right]_{t=2\gamma}^{\pi} \\
&\quad + \int_{2\gamma}^{\pi} \frac{w_x\left(\delta_0\right)}{t} dt + + \frac{1}{\delta_0}\int_0^{\delta_0}\left(\delta_0 \int_{2\gamma}^{\pi} \frac{\left|\varphi_x\left(u+t\right)\right|}{t^2} dt\right) du \\
&\quad \leq w_x\left(\delta_0\right)\left(1+\int_{2\gamma}^{\pi} \frac{1}{t} dt\right) \leq w_x\left(\delta_0\right)\left(1+\log \frac{k_r+1}{r+1}\right).
\end{aligned}
$$

Collecting our estimates we obtain desired estimate. ■

4.2 Proof of Theorem 3

If $w_x(\delta) \equiv 0$ then f is constant and our inequality is true. Thus we can suppose that $w_x(\delta) > 0$ for $\delta > 0$.

Let denote by

$$
\begin{aligned}
\Delta_\mu &= \left\{\nu : \left|S_\nu f(x) - f(x)\right| \geq \mu w_x(u)\right\} \\
\Gamma_\mu &= \left\{\nu : (\mu-1)\, G_x^{\circ} f(u)_{1,s} \leq \left|S_\nu f(x) - f(x)\right| \leq \mu w_x(u)\right\} \\
\Theta &= \left\{\mu : \Gamma_\mu \neq \varnothing\right\}
\end{aligned}
$$

the sets of integers $\nu \in [N_{m-2}+1, N_m]$ and μ, where $u = \frac{\pi}{N_{m-2}+1}$., then

$$
\begin{aligned}
H_m^{\lambda\varphi} f(x) &\leq \frac{1}{N_m+1}\sum_{\mu\in\Theta}\sum_{\nu\in\Gamma_\mu} \varphi\left(\left|S_\nu f(x) - f(x)\right|\right) \\
&\leq \frac{1}{N_m+1}\sum_{\mu\in\Theta}\sum_{\nu\in\Gamma_\mu} \varphi\left(\mu w_x(u)\right) \\
&= \frac{1}{N_m+1}\sum_{\mu\in\Theta} \left|\Gamma_\mu\right| \varphi\left(\mu w_x(u)\right) \\
&\leq \cdot\frac{1}{N_m+1}\sum_{\mu\in\Theta} \left|\Delta_{\mu-1}\right| \varphi\left(\mu w_x(u)\right).
\end{aligned}
$$

Using Theorem 1 we can compute that $\left|\Delta_{\mu-1}\right| \leq N_m \exp\left(-\frac{\mu-1}{K}\right)$, whence

$$
\begin{aligned}
H_m^{\lambda\varphi} f(x) &\leq \frac{1}{N_m+1}\sum_{\mu\in\Theta} N_m \exp\left(-\frac{\mu-1}{K}\right)\varphi\left(\mu w_x(u)\right) \\
&\ll \sum_{\mu\in\Theta} \exp\left(-\frac{\mu}{K}\right)\varphi\left(\mu w_x(u)\right).
\end{aligned}
$$

Since $\varphi \in \Phi$, we have

$$\begin{aligned}
H_m^{\lambda\varphi} f(x) &\ll \varphi(w_x(u)) \\
&\quad + \left(\sum_{n=0}^{n_0} + \sum_{n=n_0+1}^{\infty}\right) \sum_{\mu=2^n+1}^{2^{n+1}} \exp\left(-\frac{\mu}{K}\right) \varphi(\mu w_x(u)) \\
&\ll \varphi(w_x(u)) + \sum_{n=0}^{\infty} \sum_{\mu=2^n+1}^{2^{n+1}} \exp\left(-\frac{2^n}{K}\right) \varphi\left(2^{n+1} w_x(u)\right) \\
&\ll \varphi(w_x(u)) + \sum_{n=0}^{\infty} 2^n \exp\left(-\frac{2^n}{K}\right) \varphi(2^n w_x(u)) \\
&\ll \varphi(w_x(u)) + \sum_{n=0}^{n_0} 2^n \exp\left(-\frac{2^n}{K}\right) \varphi(2^n w_x(u)) \\
&\quad + \sum_{n=n_0+1}^{\infty} 2^n \exp\left(-\frac{2^n}{K}\right) \varphi(2^n w_x(u)) \\
&\ll \varphi(w_x(u))
\end{aligned}$$

with some n_0 , analogously as in.[9] p.108, and therefore our proof is complete.
■

4.3 Proof of Theorem 5

We start with the obvious inequality

$$H_{\cdot}^{\lambda\varphi} f(x) \ll \sum_{m=2}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \varphi(|S_\nu f(x) - f(x)|).$$

Using the Hölder inequality we obtain

$$H_{\cdot}^{\lambda\varphi} f(x) \ll \sum_{m=1}^{\infty} \left\{ \sum_{\nu=N_{m-2}+1}^{N_m} (\lambda_\nu)^s \right\}^{1/s} \left\{ \sum_{\nu=N_{m-2}+1}^{N_m} \varphi^q(|S_\nu f(x) - f(x)|) \right\}^{1/q}$$

with $\frac{1}{s} + \frac{1}{q} = 1$ $(s > 1)$, and by the assumption $(\lambda_\nu) \in \Lambda_s(N_m)$,we have

$$H_{\cdot}^{\lambda\varphi} f(x) \ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \left\{ \frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} \varphi^q(|S_\nu f(x) - f(x)|) \right\}^{1/q}.$$

The second assumption $\varphi \in \Phi$ also implies that $\varphi^q \in \Phi$. and therefore, by the Theorem 3,

$$
\begin{aligned}
H_{\cdot}^{\lambda\varphi} f(x) &\ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \left\{ \varphi^q \left(w_x \left(\frac{\pi}{N_{m-2}+2} \right) \right) \right\}^{1/q} \\
&\ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \varphi \left(w_x \left(\frac{\pi}{N_{m-2}+2} \right) \right).
\end{aligned}
$$

Thus our result is proved. ■

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